



TITLE:

Kostant's formula for a certain class of generalized Kac-Moody algebras II

AUTHOR(S):

NAITO, Satoshi

CITATION:

NAITO, Satoshi. Kostant's formula for a certain class of generalized Kac-Moody algebras II.
数理解析研究所講究録 1992, 778: 52-67

ISSUE DATE:

1992-03

URL:

<http://hdl.handle.net/2433/82463>

RIGHT:

Kostant's formula for a certain class of generalized Kac-Moody algebras II

By

Satoshi NAITO

Introduction.

A real $n \times n$ matrix $A = (a_{ij})_{i,j \in I}$ indexed by a set $I = \{1, 2, \dots, n\}$ is called a *GGCM* if it satisfies

(C1) either $a_{ii} = 2$ or $a_{ii} \leq 0$;

(C2) $a_{ij} \leq 0$ if $i \neq j$, and $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$;

(C3) $a_{ij} = 0$ implies $a_{ji} = 0$.

Let $\mathfrak{g}(A)$ be a *generalized Kac-Moody algebra* (GKM algebra), over the complex number field \mathbb{C} , associated to a symmetrizable GGCM $A = (a_{ij})_{i,j \in I}$, with Cartan subalgebra \mathfrak{h} , simple roots $\Pi = \{\alpha_i\}_{i \in I}$, and simple coroots $\Pi^\vee = \{\alpha_i^\vee\}_{i \in I}$. And let $\mathfrak{g}(A) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ be the triangular decomposition with $\mathfrak{n}^\pm = \sum_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$, where \mathfrak{g}_α is the root space attached to a root $\alpha \in \Delta^\pm$.

In the previous paper [4], we studied the \mathfrak{h} -module structure of the homology $H_j(\mathfrak{n}^-, L(\lambda))$ of \mathfrak{n}^- or the cohomology $H^j(\mathfrak{n}^+, L(\lambda))$ of \mathfrak{n}^+ with coefficients in the irreducible highest weight $\mathfrak{g}(A)$ -module $L(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^* = \text{Hom}_{\mathbb{C}}(\mathfrak{h}, \mathbb{C})$. (Remark that the cohomology $H^j(\mathfrak{n}^+, L(\lambda))$ used in [4] is slightly different from the usual Lie algebra cohomology.) Then, we proved

"Kostant's formula" under the following condition ($\hat{C}1$) on the

GGCM $A = (a_{ij})_{i,j \in I}$:

($\hat{C}1$) either $a_{ii} = 2$ or $a_{ii} = 0$ ($i \in I$).

Namely, we proved

Theorem A ([4]). Let $\Lambda \in P^+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0 \text{ } (i \in I), \text{ and } \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ if } a_{ii} = 2\}$. Denote by \mathcal{G} the set of all sums of distinct pairwise perpendicular elements from $\Pi^{im} := \{\alpha_i \in \Pi \mid a_{ii} \leq 0\}$. And we put $\mathcal{G}(\Lambda) := \{\lambda \in \mathcal{G} \mid (\lambda | \Lambda) = 0\}$, where $(\cdot | \cdot)$ is a standard bilinear form on \mathfrak{h}^* . Then, as \mathfrak{h} -modules ($j \geq 0$),

$$H^j(\mathfrak{n}^+, L(\Lambda)) \cong H_j(\mathfrak{n}^-, L(\Lambda)) \cong \sum_{\beta \in \mathcal{G}(\Lambda)}^{\oplus} \sum_{\substack{w \in W \\ \ell(w) = j - \text{ht}(\beta)}}^{\oplus} \mathbb{C}(w(\Lambda + \rho - \beta) - \rho),$$

where $\mathbb{C}(\mu)$ ($\mu \in \mathfrak{h}^*$) is the irreducible (one dimensional) \mathfrak{h} -module with weight μ . Here, ρ is a fixed element of \mathfrak{h}^* such that $\langle \rho, \alpha_i^\vee \rangle = (1/2) \cdot a_{ii}$ ($i \in I$), $\ell(w)$ is the length of an element w of the Weyl group W , and for $\beta = \sum_{i \in I} k_i \alpha_i$ ($k_i \in \mathbb{Z}_{\geq 0}$) $\in \mathcal{G}$, we put $\text{ht}(\beta) := \sum_{i \in I} k_i$.

In the present paper, using the idea of L. Liu [3] for Kac-Moody algebras, we extend the above result so that the nilpotent part \mathfrak{n}^+ of the Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$ is allowed to be the nilpotent part of a parabolic subalgebra containing \mathfrak{h} .

Let us explain in more detail. Let I^{re} (resp. I^{im}) be the subset $\{i \in I \mid a_{ii} = 2 \text{ (resp. } a_{ii} \leq 0)\}$ of the index set I . And let J be a subset of I^{re} . We define a submatrix A_J of A by $A_J :=$

$(a_{ij})_{i,j \in J}$, which is a generalized Cartan matrix (GCM). Note that there exists a certain subspace \mathfrak{h}_J of \mathfrak{h} , such that the triple $(\mathfrak{h}_J, \{\alpha_i|_{\mathfrak{h}_J}\}_{i \in J}, \{\alpha_i^\vee\}_{i \in J})$ is a *minimal realization* of the GCM A_J . Then, we can identify the Kac-Moody algebra $\mathfrak{g}(A_J)$ with the subalgebra \mathfrak{g}_J of $\mathfrak{g}(A)$ generated by e_i, f_i ($i \in J$), and \mathfrak{h}_J . Furthermore, $\mathfrak{g}_J = \mathfrak{h}_J \oplus \sum_{\alpha \in \Delta_J}^{\oplus} \mathfrak{g}_\alpha$, where $\Delta_J = \Delta \cap \sum_{i \in J} \mathbb{Z}\alpha_i$ is the root system of $(\mathfrak{g}_J, \mathfrak{h}_J)$. Now, we define the following subalgebras of $\mathfrak{g}(A)$:

$$\begin{aligned} \mathfrak{n}_J^+ &:= \sum_{\alpha \in \Delta_J^+}^{\oplus} \mathfrak{g}_\alpha, \quad \mathfrak{n}_J^- := \sum_{\alpha \in \Delta_J^+}^{\oplus} \mathfrak{g}_{-\alpha}, \quad \mathfrak{u}^+ := \sum_{\alpha \in \Delta^+(J)}^{\oplus} \mathfrak{g}_\alpha, \\ \mathfrak{u}^- &:= \sum_{\alpha \in \Delta^+(J)}^{\oplus} \mathfrak{g}_{-\alpha}, \quad \mathfrak{m} := \mathfrak{n}_J^- \oplus \mathfrak{h} \oplus \mathfrak{n}_J^+, \quad \mathfrak{p} := \mathfrak{m} \oplus \mathfrak{u}^+, \end{aligned}$$

where $\Delta(J) := \Delta \setminus \Delta_J$, $\Delta_J^+ = \Delta^+ \cap \Delta_J$, $\Delta^+(J) = \Delta^+ \cap \Delta(J)$. We call $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}^+$ the parabolic subalgebra of $\mathfrak{g}(A)$ defined by J . Note that since the triple $(\mathfrak{h}, \{\alpha_i\}_{i \in J}, \{\alpha_i^\vee\}_{i \in J})$ is a *realization* (but not a minimal realization) of the GCM A_J , $\mathfrak{m} = \mathfrak{g}_J + \mathfrak{h}$ can be regarded as a Kac-Moody algebra associated to A_J , whose Cartan subalgebra is \mathfrak{h} .

Recall that the Weyl group W of $\mathfrak{g}(A)$ is defined to be the subgroup of $GL(\mathfrak{h}^*)$ generated by fundamental reflections r_i ($i \in I^{\text{re}}$). Now, let W_J be the subgroup of W generated by r_i ($i \in J$), which is the Weyl group of \mathfrak{m} . And we put $W(J) := \{w \in W \mid w(\Delta^-) \cap \Delta^+ \subset \Delta^+(J)\}$ ($= \{w \in W \mid w^{-1}(\Delta_J^+) \subset \Delta^+\}$). Then, we will obtain the following theorem. (Here, as in [4], the cohomology $H^j(\mathfrak{u}^+, L(\Lambda))$ is slightly different from the usual one, whereas the homology $H_j(\mathfrak{u}^-, L(\Lambda))$ is the usual Lie algebra homology. See §3 for the

definition.)

Theorem. Let $\Lambda \in P^+$. Assume that the GGCM $A = (a_{ij})_{i,j \in I}$ is symmetrizable and satisfies the condition $(\hat{C}1)$. Then,

$$H^j(u^+, L(\Lambda)) \cong H_j(u^-, L(\Lambda)) \cong \sum_{\beta \in G(\Lambda)}^{\oplus} \sum_{\substack{w \in W(J) \\ \ell(w) = j - \text{ht}(\beta)}}^{\oplus} L_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho),$$

as \mathfrak{m} -modules ($j \geq 0$). Here, for $\mu \in P_J^+ := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ (} i \in J \text{)}\}$, $L_{\mathfrak{m}}(\mu)$ is the irreducible highest weight \mathfrak{m} -module with highest weight μ .

Note that when $J = \emptyset$, this theorem is nothing but Theorem A, since in this case, $u^+ = \mathfrak{n}^+$, $u^- = \mathfrak{n}^-$, $\mathfrak{m} = \mathfrak{h}$, and $W(J) = W$.

This paper is organized as follows. In §1, we review some basic results for GKM algebras, especially the Weyl-Kac-Borcherds character formula. In §2, we will introduce an algebra \mathcal{F} of formal \mathfrak{m} -characters, where we can carry out certain formal operations. In §3, we rewrite some results of L. Liu [3] for Kac-Moody algebras, which can be proved for GKM algebras in just the same way that they are proved for Kac-Moody algebras. In §4, we prove our main theorem stated above, combining the results of [3] and [4].

§1. The category \mathcal{O} and character formula.

In this section, we prepare fundamental results about GKM

algebras for later use. For detailed accounts of this section, see [1] and [2].

We put $I := \{1, 2, \dots, n\}$. Let $g(A)$ be the GKM algebra associated to a GGCM $A = (a_{ij})_{i,j \in I}$ with the Cartan subalgebra \mathfrak{h} .

Definition 1.1 ([2]). \mathcal{O} is the category of all \mathfrak{h} -modules V satisfying the following:

(1) V admits a weight space decomposition $V = \sum_{\lambda \in \mathcal{P}(V)}^{\oplus} V_{\lambda}$, where $\mathcal{P}(V)$ is the set of all weights of V . And each weight space V_{λ} is finite dimensional ($\lambda \in \mathcal{P}(V)$);

(2) there exist a finite number of elements $\lambda_i \in \mathfrak{h}^*$ ($1 \leq i \leq s$) such that $\mathcal{P}(V) \subset \bigcup_{i=1}^s D(\lambda_i)$, where $D(\lambda_i) := \{\lambda_i - \beta \mid \beta \in Q_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i\}$ ($1 \leq i \leq s$).

Note that the category \mathcal{O} is closed under the operations of taking submodules, quotients, finite direct sums, and finite tensor products.

Now, let \mathcal{E} be the algebra over \mathbb{C} consisting of all series of the form $\sum_{\lambda \in \mathfrak{h}^*} c_{\lambda} e(\lambda)$, where $c_{\lambda} \in \mathbb{C}$ and $c_{\lambda} = 0$ for λ outside a finite union of the sets of the form $D(\mu)$ ($\mu \in \mathfrak{h}^*$). Here, the elements $e(\lambda)$ are called *formal exponentials*. They are linearly independent and are in one-one correspondence with the elements $\lambda \in \mathfrak{h}^*$. And the multiplication of \mathcal{E} is defined by $e(\lambda) \cdot e(\mu) := e(\lambda + \mu)$ ($\lambda, \mu \in \mathfrak{h}^*$). Then, for $V = \sum_{\lambda \in \mathfrak{h}^*}^{\oplus} V_{\lambda}$ in \mathcal{O} , we define the *formal character* of V by $\text{ch } V := \sum_{\lambda \in \mathfrak{h}^*} (\dim_{\mathbb{C}} V_{\lambda}) e(\lambda) \in \mathcal{E}$. Then, we know the following character formula.

Theorem 1.1 ([1] and [2]). Assume that A is a symmetrizable GGCM. Let $(\cdot|\cdot)$ be a fixed standard bilinear form on \mathfrak{h}^* . For $\Lambda \in P^+$, we put

$$S_\Lambda := e(\Lambda + \rho) \cdot \sum_{\beta \in \mathfrak{G}(\Lambda)} (-1)^{\text{ht}(\beta)} e(-\beta), \quad R := \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{\text{mult}(\alpha)},$$

where $\text{mult}(\alpha) := \dim_{\mathbb{C}} \mathfrak{g}_\alpha$ ($\alpha \in \Delta^+$). Then,

$$e(\rho) \cdot R \cdot \text{ch } L(\Lambda) = \sum_{w \in W} (\det w) w(S_\Lambda),$$

with $w(e(\mu)) := e(w(\mu))$ ($\mu \in \mathfrak{h}^*$).

Remark 1.1. The set $\{0\} \cup \Pi^{\text{im}}$ is contained in \mathfrak{G} by definition. And, especially when A is a GCM, \mathfrak{G} consists of only one element $0 \in \mathfrak{h}^*$.

§2. The category \mathcal{O}_J and the algebra \mathcal{F} .

In this section, we explain the notion of the category \mathcal{O}_J of \mathfrak{m} -modules. And then, we introduce the algebra \mathcal{F} of "formal \mathfrak{m} -characters" of \mathfrak{m} -modules from the category \mathcal{O}_J . Note that when $J = \emptyset$, these are nothing but the category \mathcal{O} and the algebra \mathcal{E} .

From now on, we always assume that the GGCM A is symmetrizable, and that J is a subset of $I^{\text{re}} = \{i \in I \mid a_{ii} = 2\}$. We use notations in the Introduction.

Definition 2.1 ([3]). \mathcal{O}_J is the category of all \mathfrak{m} -modules M satisfying the following:

- (1) Viewed as an \mathfrak{h} -module, M is an object of the category \mathcal{O} ;
- (2) Viewed as an \mathfrak{m} -module, M is a direct sum of irreducible highest weight \mathfrak{m} -modules $L_{\mathfrak{m}}(\lambda)$ with highest weight $\lambda \in P_J^+ = \{\mu \in \mathfrak{h}^* \mid \langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ (i \in J)}\}$.

Clearly, the category \mathcal{O}_J is closed under the operations of taking submodules, quotients, and finite direct sums. Moreover, a tensor product of two modules from \mathcal{O}_J is again in the category \mathcal{O}_J , because $L_{\mathfrak{m}}(\lambda) \otimes_{\mathbb{C}} L_{\mathfrak{m}}(\mu) \in \mathcal{O}_J$ ($\lambda, \mu \in P_J^+$) by [2, Theorem 10.7. b)] (note that the modules $L_{\mathfrak{m}}(\tau)$ ($\tau \in P_J^+$) remain irreducible as \mathfrak{g}_J -modules). The main reason of our requirement that J is a subset of I^{re} comes from the fact that this theorem holds only for Kac-Moody algebras.

The following proposition plays a fundamental role in this paper.

Proposition 2.1 ([3]). For $\Lambda \in P^+$, $L(\Lambda)$ and $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$ ($j \geq 0$) are in the category \mathcal{O}_J , where $\Lambda^j u^-$ is the exterior algebra of degree j over u^- , and is an \mathfrak{m} -module by the adjoint action since $[\mathfrak{m}, u^-] \subset u^-$ ($j \geq 0$).

Now, we define a certain algebra \mathcal{F} over \mathbb{C} . The elements of \mathcal{F} are series of the form $\sum_{\lambda \in P_J^+} c_{\lambda} m(\lambda)$, where $c_{\lambda} \in \mathbb{C}$ and $c_{\lambda} = 0$ for

λ outside a finite union of the sets of the form $D(\mu)$ ($\mu \in \mathfrak{h}^*$). Here, the elements $m(\lambda)$ are called *formal \mathfrak{m} -exponentials*. They are linearly independent and are in one-one correspondence with the elements $\lambda \in P_J^+$.

For a module M in the category \mathcal{O}_J , we define the *formal \mathfrak{m} -character* $\text{ch}_{\mathfrak{m}} M$ of M by $\text{ch}_{\mathfrak{m}} M := \sum_{\lambda \in P_J^+} [M : L_{\mathfrak{m}}(\lambda)] m(\lambda)$, where $[M : L_{\mathfrak{m}}(\lambda)]$ is the "multiplicity" of $L_{\mathfrak{m}}(\lambda)$ in M (see [2, Ch.9, Lemma 9.6]). Note that $[M : L_{\mathfrak{m}}(\lambda)]$ ($\lambda \in P_J^+$) is finite since M is in the category \mathcal{O} as an \mathfrak{h} -module. Therefore, $\text{ch}_{\mathfrak{m}} M$ is an element of the algebra \mathcal{F} for $M \in \mathcal{O}_J$. Then, the multiplication of \mathcal{F} is defined as follows: for $\lambda, \mu \in P_J^+$, $m(\lambda) \cdot m(\mu) := \text{ch}_{\mathfrak{m}}(L_{\mathfrak{m}}(\lambda) \otimes_{\mathbb{C}} L_{\mathfrak{m}}(\mu))$. Thus, \mathcal{F} becomes a commutative associative algebra over \mathbb{C} .

Following [3], we now define an algebra homomorphism $\Psi(\mathfrak{m}, \mathfrak{h}): \mathcal{F} \rightarrow \mathcal{E}$, by $\Psi(\mathfrak{m}, \mathfrak{h})(m(\lambda)) := \text{ch } L_{\mathfrak{m}}(\lambda) \in \mathcal{E}$ ($\lambda \in P_J^+$). Then, we have

Lemma 2.1. The mapping $\Psi(\mathfrak{m}, \mathfrak{h}): \mathcal{F} \rightarrow \mathcal{E}$ is injective.

Proof (cf. [3]). Let $\sum_{\lambda \in P_J^+} c_{\lambda} m(\lambda)$ be a non-zero element of \mathcal{F} . Then, there exist $\mu_i \in \mathfrak{h}^*$ ($1 \leq i \leq s$), such that $\{\lambda \in P_J^+ \mid c_{\lambda} \neq 0\} \subset \bigcup_{i=1}^s D(\mu_i)$. By replacing the set $\{\mu_i\}_{i=1}^s$ by a suitable finite subset $\{\mu'_i\}_{i=1}^t$ of \mathfrak{h}^* if necessary, we can assume that $\mu'_k - \mu'_l \notin Q = \sum_{i \in I} \mathbb{Z} \alpha_i$ ($1 \leq k \neq l \leq t$). Consider the subset $\{\text{ht}(\mu'_i - \lambda) \mid \lambda \in P_J^+ (c_{\lambda} \neq 0) \text{ and } \lambda \in D(\mu'_i) (1 \leq i \leq t)\}$ of $\mathbb{Z}_{\geq 0}$, and take $\lambda_0 \in P_J^+$ which attains the minimum of this subset. Then, clearly λ_0 is not a weight of $L_{\mathfrak{m}}(\lambda)$ ($\lambda \in P_J^+ \setminus \{\lambda_0\}$). Hence, $\Psi(\mathfrak{m}, \mathfrak{h})(\sum_{\lambda \in P_J^+} c_{\lambda} m(\lambda)) \neq 0$

$\in \mathfrak{g}$. Thus we have shown the injectivity of $\Psi(\mathfrak{m}, \mathfrak{h})$ □

§3. Some results of L. Liu.

In this section, we rewrite, in the case of GKM algebras, some of Liu's results on \mathfrak{m} -modules $H_j(u^-, L(\lambda))$ and $H^j(u^+, L(\lambda))$ ($j \geq 0$) for Kac-Moody algebras. His proofs for these results require no modifications. For details, see [3].

The homology $H_j(u^-, L(\lambda))$ of u^- with coefficients in $L(\lambda)$ ($\lambda \in \mathfrak{h}^*$) is defined as the homology of the \mathfrak{m} -module complex $\{(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\lambda), d_j\}$, where the action of \mathfrak{m} and the boundary operator d_j are defined in a usual way. The cohomology $H^j(u^+, L(\lambda))$ of u^+ with coefficients in $L(\lambda)$ is defined as the cohomology of the \mathfrak{m} -module complex $\{\text{Hom}_{\mathbb{C}}^{\mathbb{C}}(\Lambda^j u^+, L(\lambda)), d^j\}$, where the action of \mathfrak{m} and the coboundary operator d^j are usual ones. Here, for \mathfrak{h} -diagonalizable modules $V = \sum_{\mu \in \mathfrak{h}^*}^{\oplus} V_{\mu}$ and W with finite dimensional weight spaces, we put $\text{Hom}_{\mathbb{C}}^{\mathbb{C}}(V, W) := \{f \in \text{Hom}_{\mathbb{C}}(V, W) \mid f(V_{\mu}) = 0 \text{ for all but finitely many weights } \mu \in \mathfrak{h}^* \text{ of } V\}$. Note that this cohomology $H^j(u^+, L(\lambda))$ of u^+ is different from the usual one, since we have used $\text{Hom}_{\mathbb{C}}^{\mathbb{C}}(\Lambda^j u^+, L(\lambda))$ instead of $\text{Hom}_{\mathbb{C}}(\Lambda^j u^+, L(\lambda))$ as the space of j cochains ($j \geq 0$) (see also [3]).

Then, we have the following, due to L. Liu.

Proposition 3.1 ([3]). For any $\Lambda \in P^+$ and $j \in \mathbb{Z}_{\geq 0}$, $H^j(u^+, L(\Lambda))$ is isomorphic to $H_j(u^-, L(\Lambda))$ as \mathfrak{m} -modules.

So, from now on, we concentrate on \mathfrak{m} -modules $H_j(u^-, L(\Lambda))$ ($j \geq 0$). Since $L(\Lambda)$ and $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$ are in the category \mathcal{O}_J by Proposition 2.1, $H_j(u^-, L(\Lambda))$ is also in \mathcal{O}_J , and so is a direct sum of $L_{\mathfrak{m}}(\mu)$ ($\mu \in P_J^+$) as \mathfrak{m} -modules. Furthermore, we have

Proposition 3.2 ([3]). Let $(\cdot | \cdot)$ be a fixed standard bilinear form on \mathfrak{h}^* . Then, for any $\Lambda \in P^+$ and $j \in \mathbb{Z}_{\geq 0}$, every \mathfrak{m} -irreducible component of $H_j(u^-, L(\Lambda))$ is of the form $L_{\mathfrak{m}}(\mu)$ ($\mu \in P_J^+$) with $(\mu + \rho | \mu + \rho) = (\Lambda + \rho | \Lambda + \rho)$.

§4. Kostant's formula for GKM algebras.

In this section, we prove "Kostant's formula" for GKM algebras, which is a generalization of that in my previous paper [4]. Here, we assume that the symmetrizable GGCM $A = (a_{ij})_{i,j \in I}$ satisfies the following condition ($\hat{C}1$):

($\hat{C}1$) either $a_{ii} = 2$ or $a_{ii} = 0$ ($i \in I$).

And recall that J is a subset of I^{re} .

4.1. Necessity condition. Now, we review some results given in [4, Lemma 4.2] and its proof. Let $(\cdot | \cdot)$ be a standard bilinear form on \mathfrak{h}^* . Then, we have

Lemma 4.1 ([4]). Let $\Lambda \in P^+$. If, for some j ($j \geq 0$), μ is a weight of $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$ and satisfies $(\mu + \rho | \mu + \rho) = (\Lambda + \rho | \Lambda + \rho)$, then

- (1) there exist a $\beta_0 \in \mathfrak{G}(\Lambda)$ and a $w_0 \in W$, such that $\ell(w_0) + \text{ht}(\beta_0) = j$ and $\mu = w_0(\Lambda + \rho - \beta_0) - \rho$;
- (2) the multiplicity of μ in $(\Lambda \pi^-) \otimes_{\mathbb{C}} L(\Lambda)$ is equal to one, where $\Lambda \pi^- = \sum_{j \geq 0}^{\oplus} \Lambda^j \pi^-$.

Let us fix $\Lambda \in P^+$. From the above, we can prove the following.

Lemma 4.2. Assume that $\mu \in \mathfrak{h}^*$ is a weight of $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$ for some $j \in \mathbb{Z}_{\geq 0}$, and satisfies $(\mu + \rho | \mu + \rho) = (\Lambda + \rho | \Lambda + \rho)$. Then,

- (a) there exist a $\beta \in \mathfrak{G}(\Lambda)$ and a $w \in W(J)$, such that $\ell(w) + \text{ht}(\beta) = j$ and $\mu = w(\Lambda + \rho - \beta) - \rho$;
- (b) the multiplicity of μ in $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$ is equal to one.

Proof. If $\mu \in \mathfrak{h}^*$ is a weight of $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$, then μ is a weight of $(\Lambda^j \pi^-) \otimes_{\mathbb{C}} L(\Lambda)$, since $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$ can be regarded as a submodule of $(\Lambda^j \pi^-) \otimes_{\mathbb{C}} L(\Lambda)$. Then, by Lemma 4.1, it follows that there exist a $\beta_0 \in \mathfrak{G}(\Lambda)$ and a $w_0 \in W$, such that $\ell(w_0) + \text{ht}(\beta_0) = j$ and $\mu = w_0(\Lambda + \rho - \beta_0) - \rho$, and that the multiplicity of μ in $(\Lambda \pi^-) \otimes_{\mathbb{C}} L(\Lambda)$ is equal to one. So, we have only to show that $w_0 \in W(J) = \{w \in W \mid w(\Delta^-) \cap \Delta^+ \subset \Delta^+(J)\}$. Now, recall that $w_0(\rho) - \rho = - \sum_{\alpha \in \Phi_{w_0}} \alpha$, where $\Phi_{w_0} = w_0(\Delta^-) \cap \Delta^+$ (see [4, Proposition 1.2.b])). Express $\beta_0 = \sum_{k=1}^m \alpha_{i_k}$, where $m = \text{ht}(\beta_0)$, $\alpha_{i_k} \in \Pi^{im}$ ($1 \leq k \leq m$), and $i_r \neq i_t$ ($1 \leq r \neq t \leq m$). And take non-zero root vectors $E_k \in \mathfrak{g}_{-w_0(\alpha_{i_k})}$ ($1 \leq k \leq m$), $E_{\alpha} \in \mathfrak{g}_{-\alpha}$ ($\alpha \in \Phi_{w_0}$), and a non-zero weight vector $v \in L(\Lambda)_{w(\Lambda)}$. Then, it is clear that

$0 \neq (E_1 \wedge \cdots \wedge E_m) \wedge (\bigwedge_{\alpha \in \Phi_{w_0}} E_\alpha) \otimes v \in (\Lambda \pi^-) \otimes_{\mathbb{C}} L(\Lambda)$ is a weight vector of weight μ (cf. the proof of [4, Lemma 4.2]). Since the multiplicity of μ in $(\Lambda \pi^-) \otimes_{\mathbb{C}} L(\Lambda)$ is equal to one, and μ is a weight of $(\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$ by assumption, it follows that $(E_1 \wedge \cdots \wedge E_m) \wedge (\bigwedge_{\alpha \in \Phi_{w_0}} E_\alpha) \otimes v \in (\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)$. Therefore, $\alpha \in \Delta^+(J)$ ($\alpha \in \Phi_{w_0}$). Hence, $w_0 \in W(J)$ by definition of $W(J)$. Thus we have proved Lemma 4.2. \square

By Proposition 3.2 and Lemma 4.2, we have the following.

Proposition 4.1. Let $j \in \mathbb{Z}_{\geq 0}$. If $L_{\mathfrak{m}}(\mu)$ ($\mu \in P_J^+$) is an \mathfrak{m} -irreducible component of $H_j(u^-, L(\Lambda))$, then

(a) $\mu = w(\Lambda + \rho - \beta) - \rho$, for some $\beta \in \mathfrak{G}(\Lambda)$ and some $w \in W(J)$, such that $\ell(w) + \text{ht}(\beta) = j$;

(b) $L_{\mathfrak{m}}(\mu)$ occurs with multiplicity one as \mathfrak{m} -irreducible components of $H_j(u^-, L(\Lambda))$.

4.2. Sufficiency condition. Here, we use the setting in §2. Let $\Lambda \in P^+$. Before carrying out formal operations on formal \mathfrak{m} -characters in the algebra \mathcal{F} , we note that $w(\Lambda + \rho - \beta) - \rho$ differs if $w \in W$ or $\beta \in \mathfrak{G}$ differs (see the proof of [4, Proposition 4.2]).

Lemma 4.3. For $w \in W(J)$ and $\beta \in \mathfrak{G}$, $w(\Lambda + \rho - \beta) - \rho \in P_J^+$.

Proof. We have to show that $\langle w(\Lambda + \rho - \beta) - \rho, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for $i \in J$. Since $w \in W(J)$ and $i \in J \subset I^{\text{re}}$, it follows that $w^{-1}(\alpha_i) \in \Delta^+$ since $W(J) = \{w \in W \mid w^{-1}(\Delta_J^+) \subset \Delta^+\}$. So, we have $w^{-1}(\alpha_i^\vee) \in (\Delta^\vee)^+$, where $\Delta^\vee = \Delta({}^t A) \subset \mathfrak{h}$ is the dual root system of $\mathfrak{g}(A)$ (see [2]). Moreover, $w^{-1}(\alpha_i^\vee) \in \sum_{j \in I^{\text{re}}} \mathbb{Z} \alpha_j^\vee$ since $J \subset I^{\text{re}}$. On the other hand, we have

$$\begin{aligned} \langle w(\Lambda + \rho - \beta) - \rho, \alpha_i^\vee \rangle &= \langle \Lambda + \rho - \beta, w^{-1}(\alpha_i^\vee) \rangle - \langle \rho, \alpha_i^\vee \rangle \\ &= \langle \Lambda, w^{-1}(\alpha_i^\vee) \rangle - \langle \beta, w^{-1}(\alpha_i^\vee) \rangle + \langle \rho, w^{-1}(\alpha_i^\vee) \rangle - 1 \end{aligned}$$

Since $\Lambda \in P^+$ and β is a sum of elements from Π^{im} , we deduce that $\langle w(\Lambda + \rho - \beta) - \rho, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ from the above equality. Thus the assertion has been proved. \square

Proposition 4.2. For $\Lambda \in P^+$, there holds in the algebra \mathcal{F} ,

$$\begin{aligned} \sum_{j \geq 0} (-1)^j \text{ch}_{\mathfrak{m}}(H_j(u^-, L(\Lambda))) &= \\ &= \sum_{\beta \in \mathcal{G}(\Lambda)} (-1)^{\text{ht}(\beta)} \sum_{w \in W(J)} (\det w) m(w(\Lambda + \rho - \beta) - \rho). \end{aligned}$$

Proof. Both sides of the above equality are clearly in the algebra \mathcal{F} by Lemma 4.3. So, because $\Psi(\mathfrak{m}, \mathfrak{h}): \mathcal{F} \rightarrow \mathcal{E}$ is injective, we have only to show the following in the algebra \mathcal{E} (cf. also Proposition 4.1).

$$\begin{aligned}
(\#) \quad \sum_{j \geq 0} (-1)^j \text{ch}(H_j(u^-, L(\Lambda))) &= \\
&= \sum_{\beta \in \mathfrak{G}(\Lambda)} (-1)^{\text{ht}(\beta)} \sum_{w \in W(J)} (\det w) \cdot \text{ch } L_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho).
\end{aligned}$$

By the well-known Euler-Poincaré principle, the left hand side of (#) is equal to

$$\begin{aligned}
\sum_{j \geq 0} (-1)^j \text{ch}(H_j(u^-, L(\Lambda))) &= \sum_{j \geq 0} (-1)^j \text{ch}((\Lambda^j u^-) \otimes_{\mathbb{C}} L(\Lambda)) = \\
&= (\sum_{j \geq 0} (-1)^j \cdot \text{ch } \Lambda^j u^-) \cdot \text{ch } L(\Lambda) = \prod_{\alpha \in \Delta^+(J)} (1 - e(-\alpha))^{\text{mult}(\alpha)} \cdot \text{ch } L(\Lambda) = \\
&= \frac{e(\rho) \cdot \prod_{\alpha \in \Delta^+} (1 - e(-\alpha))^{\text{mult}(\alpha)}}{e(\rho) \cdot \prod_{\alpha \in \Delta_J^+} (1 - e(-\alpha))^{\text{mult}(\alpha)}} \cdot \text{ch } L(\Lambda).
\end{aligned}$$

By Theorem 1.1, this is equal to

$$e(-\rho) \cdot R_J^{-1} \cdot \sum_{w \in W} (\det w) \sum_{\beta \in \mathfrak{G}(\Lambda)} (-1)^{\text{ht}(\beta)} e(w(\Lambda + \rho - \beta)),$$

$$\text{where } R_J := \prod_{\alpha \in \Delta_J^+} (1 - e(-\alpha))^{\text{mult}(\alpha)}.$$

On the other hand, by Theorem 1.1 applied for an $\mathfrak{m} (= \mathfrak{g}_J + \mathfrak{h})$ -module $L_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho)$, the right hand side of (#) is equal to

$$e(-\rho) \cdot R_J^{-1} \cdot \sum_{\beta \in \mathfrak{G}(\Lambda)} (-1)^{\text{ht}(\beta)} \sum_{w \in W(J)} (\det w) \sum_{u \in W_J} (\det u) e(u(w(\Lambda + \rho - \beta)))$$

$$= e(-\rho) \cdot R_J^{-1} \cdot \sum_{\beta \in \mathcal{G}(\Lambda)} (-1)^{\text{ht}(\beta)} \sum_{w \in W(J), u \in W_J} (\det uw) e(uw(\Lambda + \rho - \beta)).$$

Now, we quote the fact that every $w \in W$ can be uniquely expressed in the form $w_J \cdot w(J)$, where $w_J \in W_J$ and $w(J) \in W(J)$. Note that this fact requires J to be a subset of I^{re} . (See [3] for the proof.) Therefore, the above is equal to

$$\begin{aligned} & e(-\rho) \cdot R_J^{-1} \cdot \sum_{\beta \in \mathcal{G}(\Lambda)} (-1)^{\text{ht}(\beta)} \sum_{w \in W} (\det w) e(w(\Lambda + \rho - \beta)) \\ &= e(-\rho) \cdot R_J^{-1} \cdot \sum_{w \in W} (\det w) \sum_{\beta \in \mathcal{G}(\Lambda)} (-1)^{\text{ht}(\beta)} e(w(\Lambda + \rho - \beta)). \end{aligned}$$

Thus, we have proved the equality (#). This completes the proof of Proposition 4.2. \square

By Propositions 4.1 and 4.2, we have the following.

Proposition 4.3. Fix $j \in \mathbb{Z}_{\geq 0}$. And put $\mu := w(\Lambda + \rho - \beta) - \rho$, where $\beta \in \mathcal{G}(\Lambda)$ and $w \in W(J)$, such that $\ell(w) + \text{ht}(\beta) = j$. Then, $L_{\mathfrak{m}}(\mu)$ occurs as \mathfrak{m} -irreducible components of $H_j(u^-, L(\Lambda))$.

Summarizing Propositions 3.1, 4.1, and 4.3, we obtain the following theorem.

Theorem 4.1 (Kostant's formula). Let $\Lambda \in P^+$. And let $\mathfrak{g}(\Lambda)$ be the GKM algebra associated to a symmetrizable GGCM $A = (a_{ij})_{i,j \in I}$ satisfying (C1). We assume that the subset J of I is

contained in $I^{re} = \{i \in I \mid a_{ii} = 2\}$. Then, as \mathfrak{m} -modules ($j \geq 0$),

$$H^j(u^+, L(\Lambda)) \cong H_j(u^-, L(\Lambda))$$

$$\cong \sum_{\beta \in \mathfrak{G}(\Lambda)}^{\oplus} \sum_{\substack{w \in W(J) \\ \ell(w) = j - \text{ht}(\beta)}}^{\oplus} L_{\mathfrak{m}}(w(\Lambda + \rho - \beta) - \rho).$$

Here, $L_{\mathfrak{m}}(\mu)$ ($\mu \in P_J^+$) is the irreducible highest weight \mathfrak{m} -module with highest weight μ .

Remark 4.1. In our arguments, the assumption that J is a subset of I^{re} plays an essential role. So, we can not remove it.

Department of Mathematics
Faculty of Science
Kyoto University

References

- [1] R. Borcherds, Generalized Kac-Moody algebras, J. Algebra, 115(1988), 501-512.
- [2] V. G. Kac, Infinite dimensional Lie algebras (3rd edition), Cambridge University Press, Cambridge, 1990.
- [3] L. Liu, Kostant's formula for Kac-Moody Lie algebras, Yale University dissertation, 1990.
- [4] S. Naito, Kostant's formula for a certain class of generalized Kac-Moody algebras, to appear.